

國立政治大學應用數學系九十學年度第二學期研究生學科考試試題

科目：離散數學

Qualifying Examination for Discrete Mathematics

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\* \* \* Among the 6 problems, do any 5 of them \* \* \*

1. For any plane graph depiction of a connected planar graph, the Euler's formula is that  $r = e - v + 2$ , where  $r$  is the number of regions,  $e$  the number of edges and  $v$  the number of vertices. Use Euler's formula to prove that  $K_5$  and  $K_{3,3}$  are non-planar.

2. Evaluate  $\binom{n}{0} - 2\binom{n}{1} + 3\binom{n}{2} + \cdots + (-1)^n(n+1)\binom{n}{n}$ .

3. How many ways are there to distribute 25 identical balls into seven distinct boxes if the first box can have no more than 10 balls but amount can go into each of the other six boxes?

4. Solve the following recurrence relations when  $a_0 = 1$ :

- (a)  $a_n^2 = 2a_{n-1}^2 + 1$ . (Hint: Let  $b_n = a_n^2$ .)

- (b)  $a_n = -na_{n-1} + n!$ . (Hint: Define an appropriate  $b_n$  as in part (a).)

5. How many different integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20, 0 \leq x_i \leq 8 \text{ for } 1 \leq i \leq 6.$$

6. A baton is painted with equal-sized cylindrical bands. Each band can be painted black or white. If the baton is unoriented as when spun in the air, how many different 2-colorings of the baton are possible if the baton has 2 bands? 3 bands?  $n$  bands?

國立政治大學應用數學系九十一學年度第一學期研究生學科考試試題

科目：離散數學

請選五題作答，每題二十分。如果未能選擇，一律以第一題至第五題為準。

1. 一年愛班有30個小朋友，班級家長會時，只有15位小朋友的家長參加，因為是隨意入座，結果竟然沒有家長坐對位置，請問這種情形共有多少呢？
2. 應數所新生座談時，只有8位新同學出席，經過自我介紹後，所代統計發現第一位同學只與一位同學互相認識，第二位同學也是只與一位同學互相認識，第三位同學與二位同學互相認識，第四位與三位互相認識，第五位與四位互相認識，第六位與五位互相認識，第七位與六位互相認識，第八位與七位互相認識，系主任看了後，認為統計有誤，請問是所代正確還是系主任正確？為什麼？  
(註：如果只有甲認識乙，而乙不認識甲，就不是互相認識。)
3. 高雄市長人選協商未果，造成國民黨與親民黨之間小尷尬，某次聚餐採國民黨、親民黨和無黨籍人士自由參加，但是國民黨與親民黨都不願意相鄰而坐，如果有一排20個的座位，共有幾種不同的排座位法？

4. 我們知道

$$C_0^n = 1; C_k^n = \frac{n(n-1)\cdots(n-k+1)}{k!}, k > 0$$

中可以將 $n$ 推廣為有理數。證明：

$$\frac{1}{\sqrt{1+x}} = \sum_{k \geq 0} C_k^{-\frac{1}{2}} x^k.$$

5. 圓上有點 $P_1, P_2, \dots, P_{20}$ ，利用這20個點畫10條弦，使得每兩條弦都不相交，而且每個點都用到，如果說這樣的畫法是此圓的“拼圖”，請問共有幾種“拼圖”？
6. 離散數學的教科書有六個面，用綠、藍和橘三種顏色來塗色，共有幾種不同的塗法？

國立政治大學應用數學系九十一學年度第二學期學科考試試題

科目：離散數學

1. 應用數學系大一有新生40名，數學相關課程共有微積分、離散數學和統計學三門，系主任爲了避免學生剛進來就二一，限制每位同學最多選一門數學相關課程，共有幾種選法？
2. 將一個2公分×2公分×3公分的長方體中六個面，塗上紅、白或黑三種顏色，共有幾種塗法？
3. 證明任意六個人中，必有三個人兩兩互相認識或三個人兩兩不互相認識。（如果只有甲認識乙，而乙不認識甲，則甲和乙不互相認識。）
4. 文山區圍棋賽共有號次爲1, 2, 3, 4, 5, 6, 7, 8, 9等九人參加，採單淘汰制，每輪比賽前先將未淘汰的號次由小到大排列，取相鄰的號次對奕，沒有對手的號次自動晉級下一輪，直到冠軍產生爲止，稱此爲「競賽過程」，共有幾種不同的「競賽過程」？
5. 情報局內每筆資料都用一個0, 1, 2, 3所構成的字串來傳送，爲了安全上的理由，局內規定每個字串的長度皆相同且0出現偶數次，今有一千筆資料要傳送，問字串的長度至少要多長呢？
6. 用組合的方法（combinatorial method）證明

$$\sum_{i=0}^n C_i^{\frac{1}{2}} C_{n-i}^{\frac{1}{2}} = 0, \forall n \geq 2,$$

定義

$$C_k^{\frac{1}{2}} = \begin{cases} 1, & k = 0 \\ \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - k + 1)}{k!}, & k \geq 1 \end{cases}$$

國立政治大學應用數學系九十二學年度第一學期研究生學科考試試題

科目：離散數學

1. A line graph  $L(G)$  of a graph  $G$  has a vertex of  $L(G)$  for each edge of  $G$  and an edge of  $L(G)$  joining each pair of vertices corresponding to two edges in  $G$  with a common end vertex.
  - (a) Show that  $L(K_5)$  is nonplanar.
  - (b) Find a planar graph whose line graph is nonplanar.
2. Show by a combinatorial argument that
  - (a)  $(n-r) \binom{n+r-1}{r} \binom{n}{r} = n \binom{n+r-1}{2r} \binom{2r}{r}$ .
  - (b)  $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$ .
3.
  - (a) Find a generating function for  $a_n$ , the number of partitions that add up to at most  $n$ .
  - (b) Find a generating function for  $a_n$ , the number of partitions of  $n$  into three parts in which no part is larger than the sum of the other two.
  - (c) Find a generating function for  $a_n$ , the number of different (incongruent) triangles with integral sides and perimeter  $n$ .
4.
  - (a) Find and solve a recurrence relation for the number of different square subboards of any size that can be drawn on an  $n \times n$  chessboard.
  - (b) Repeat part (a) for rectangular subboards of any size.
5. How many ways are there to distribute 10 books to 10 children (one to a child) and then collect the books and redistribute them with each child getting a new one?
6.
  - (a) Find the number of  $2 \times 4$  chessboards distinct under rotation whose squares are colored red or black.
  - (b) Suppose that two chessboards are also considered equivalent (aside from rotation symmetry) if one can be obtained from the other by complementing red and black colors. How many different  $2 \times 4$  boards are there?

以上題目請選5題作答，每題20分。

國立政治大學應用數學系九十二學年度第二學期研究生學科考試試題

科目：離散數學

1. Prove that every  $n$ -vertex plane graph isomorphic to its dual has  $2n - 2$  edges. For all  $n \geq 4$ , construct a simple  $n$ -vertex plane graph isomorphic to its dual.
2. Give a combinatorial argument to evaluate
  - (a)  $\sum_{k=0}^n \binom{n}{k} \binom{m}{k}, n \leq m.$
  - (b)  $\sum_{k=0}^m \binom{n+k}{k}.$
3. How many  $r$ -digit quaternary sequences (whose digits are 0, 1, 2, and 3) are there in which the total number of 0s and 1s is even?
4. Find and solve a recurrence relation for the number of ways to make a pile of  $n$  chips using red, white, and blue chips and such that no two red chips are together.
5. How many ways are there to distribute  $r$  distinct objects into  $n$  indistinguishable boxes with no box empty?
6. How many 4-bead necklaces are there in which each bead is one of the colors  $b$ ,  $r$ , or  $p$ , and there is at least one  $p$ ?

國立政治大學應用數學系九十三學年度第一學期研究生學科考試試題

科目：離散數學

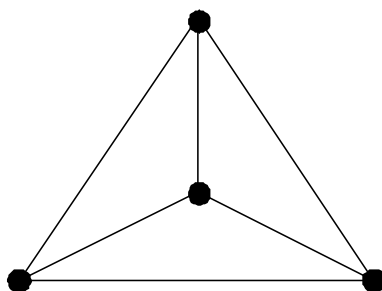
1. (a) List all nonisomorphic undirected graphs with four vertices.  
(b) If a connected planar graph with  $n$  vertices all of degree 4 has 10 regions, determine  $n$ .
2. (a) Show by a combinatorial argument that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

- (b) Give a combinatorial argument to evaluate

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k}, \quad n \leq m.$$

3. Find the number of  $r$ -digit quaternary sequences (whose digits are 0, 1, 2, and 3) with an even number of 0s and an odd number of 1s.
4. Find a recurrence relation and solve it with generating functions for the number of ways to divide an  $n$ -gon into triangles with noncrossing diagonals.
5. There are 15 students, three (distinct) students each from 5 different high schools. There are 5 admissions officers, one from each of 5 colleges. Each of the officers successively pick 3 of the students to go to their college. How many ways are there to do this so that no officer picks 3 students from the same high school?
6. (a) Find the number of different  $m$ -colorings of the vertices of the following floating figure.  
(Note that “floating” means that all possible rotations and reflection are allowed.)



- (b) Find the number of different  $m$ -colorings of the edges of the above floating figure.

國立政治大學應用數學系九十三學年度第二學期研究生學科考試試題

科目：離散數學

1. Prove that a self-complementary (a graph isomorphic to its complement) graph with  $n$  vertices exists if and only if  $n$  or  $n - 1$  is divisible by 4. (Hint: When  $n$  is divisible by 4, generalize the structure of  $P_4$  (a path with 4 vertices) by splitting the vertices into 4 groups. For  $n \equiv 1 \pmod{4}$ , add one vertex to the graph constructed for  $n - 1$ .)

2. (a) Give a combinatorial argument to evaluate the sum

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + i\binom{n}{i} + \cdots + n\binom{n}{n}.$$

- (b) If we write down all the combinations of  $n$  letters  $\{1, 2, \dots, n\}$ , that is, all 1-combinations, all 2-combinations, ..., how many times does each letter occur?
3. Find the exponential generating functions for the number of ways to distribute  $r$  distinct objects into five different boxes when  $b_1 < b_2 \leq 4$ , where  $b_1, b_2$  are the numbers of objects in boxes 1 and 2, respectively.
4. Find and solve a recurrence relation for the number of  $n$ -digit ternary sequences in which no 1 appears to the right of any 2.
5. Each of  $n$  gentlemen checks both a hat and an umbrella. The hats are returned at random and the umbrellas are independently returned at random. What is the probability that no man gets back both his hat and umbrella?
6. How many four-bead necklaces are there in which each bead is one of the colors  $b$ ,  $r$ , or  $p$ , and there is at least one  $p$ ?

國立政治大學應用數學系九十四學年度第一學期研究生學科考試試題

科目：離散數學

1. Show that the complete bipartite graph  $K_{m,n}$  is not planar for  $m \geq 3$  and  $n \geq 3$ .
2. For any positive integers  $m$  and  $n$ , show that  $(m!)^{n+1}$  divides  $(mn)!$ .
3. How many ways are there to choose some integer(s) from 1 to 100 without consecutive integers?
4. How many ways are there to write down  $n$  pairs of parentheses?
5. Show that the number of elements in exactly 2 sets of  $A_1, A_2, \dots, A_n$  is  $\sum_{k=2}^{n-2} (-1)^k \binom{n}{k} S_k$  where  $S_k$  denotes the sum of the sizes of all  $k$ -tuple intersections of  $A_1, A_2, \dots, A_n$ .
6. How many ways are there to color the six faces of a rectangular box with a square base using three colors?



國立政治大學應用數學系九十四學年度第二學期研究生學科考試試題

科目：離散數學

1. Let  $G$  be a connected graph. Show that  $G$  has an Euler cycle if and only if each vertex of  $G$  has even degree.
2. 證明  $\sum_{k=-n}^m \binom{m-k}{r} \binom{n+k}{s} = \binom{m+n+1}{r+s+1}$ .
3. 圓上有  $n$  個點，用來畫出最多條弦，使得任意兩條弦在圓內皆不相交，形成一組「最大弦分割」，請問共有幾組「最大弦分割」？
4. 由  $\{0, 1, 2\}$  所構成的三元數列，長度為  $n$  且 0 不能相鄰，請問共有多少個數列？
5. 在集合  $\{1, 2, 3, \dots, 10\}$  上，可以定出多少個等價關係 (equivalence relation)，恰好有 5 個等價班 (equivalence class) ？
6. 用紅、黃、藍三種珠子，來串出一個 6 顆珠子的手環，請問共有幾種串法？

國立政治大學應用數學系九十五學年度第一學期研究生學科考試試題

科目：離散數學

1. Given any planar graph  $G$  with  $r$  regions,  $e$  edges,  $v$  vertices, and  $c$  components, show that  $r - e + v - c = 1$ .
2. Draw  $n$  lines on a plane, find the maximum number of bounded regions.
3. How many  $n$ -digit sequences from  $\{0, 1, 2, 3\}$  are there with an even number of 0's and an even number of 1's?
4. Show that  $\sum_{n=0}^m \binom{2n}{n} \binom{2m-2n}{m-n} = 4^m$ .
5. How many ways are there to partition  $\{1, 2, \dots, n\}$  into exactly  $k$  subsets?
6. Find the number of ways to color the 12 edges of a cube using 2 colors.

國立政治大學應用數學系九十五學年度第二學期研究生學科考試試題

科目：離散數學

1. Show that every forest of  $n$  vertices and  $k$  components has  $n - k$  edges.
2. Show that  $(n!)^{n+1}$  divides  $(n^2)!$ .
3. How many ways are there to distribute 10 different books among 5 students such that each student gets at least one book?
4. Find the number of  $n$ -digit ternary sequences in which no 1 appears to the right of any 2.
5. Find the number of ways to divide an  $n$ -gon into triangles with noncrossing diagonals.
6. How many ways are there to color the vertices of a regular 6-gon using three colors?

國立政治大學應用數學系九十六學年度第一學期研究生學科考試試題

科目：離散數學

1. Show that if  $G$  is a connected planar graph, then any plane graph depiction of  $G$  has  $r = e - v + 2$  regions where  $e$  is the number of edges and  $v$  is the number of vertices.

2. Show that

$$\sum_{k=0}^m \binom{m}{k} \binom{n}{r+k} = \binom{m+n}{m+r}.$$

3. Use a recurrence relation to find the number of  $r$ -digit quaternary sequences with an even number of 0's.

4. Use a generating function to find the number of  $r$ -digit quaternary sequences with an even number of 0's.

5. Let  $A_1, A_2, \dots, A_n$  be  $n$  sets in a universe  $U$  of  $N$  elements and  $N_m$  be the number of elements in exactly  $m$  sets. Show that

$$N_m = \sum_{k=0}^{n-m} (-1)^k \binom{m+k}{m} S_{m+k}$$

where  $S_{m+k}$  is the sum of sizes of all  $(m+k)$ -tuple intersections of the  $A_i$ 's.

6. How many ways are there to color six faces of a cube using  $r$  colors?