國立政治大學應用數學系九十二學年度第二學期研究生學科考試試題

科目：微分方程式

(1) Let $f : \mathbb{R} \to \mathbb{R}$ a bounded and continuous function. For each $n = 1, 2, \ldots$, let $x_n : [0, 1] \to \mathbb{R}$ be a solution of the differential equation

$$x' = f(x). \tag{$*$}$$

If the sequence $\{x_n(0)\}$ converges, prove that there exists a subsequence of $\{x_n\}$ which converges uniformly to a solution of $(*)$ on $[0, 1]$.

(2) Let Q_1 and Q_2 be two continuous functions on an interval $[a, b]$ with $Q_2(t) > Q_1(t)$ for any $t \in [a, b]$. Let $\varphi_1(t)$ be a solution of

$$x'' + Q_1(t)x = 0$$

and $\varphi_2(t)$ be a solution of

$$x'' + Q_2(t)x = 0$$

on $a \le t \le b$. If t_1 and t_2 are consecutive zeros of φ_1 on $[a, b]$, show that φ_2 has at least one zero at some point of the open interval $t_1 < t < t_2$.

(3) Find all critical points of the nonlinear differential system

$$\begin{cases} u_1' = 1 - u_1 u_2 \\ u_2' = u_1 - u_2^3 \end{cases}$$

and determine whether they are stable or unstable.

(4) For each of the following systems construct a Lyapunov function of the form $c_1 u_1^2 + c_2 u_2^2$ to determine whether the trivial solution is stable, asymptotically stable or unstable.

$$\text{(a)} \begin{cases} u_1' = -u_1 + e^{u_1} u_2 \\ u_2' = -e^{u_1} u_1 - u_2 \end{cases} \qquad \text{(b)} \begin{cases} u_1' = -u_1^3 + u_1^2 u_2 \\ u_2' = -u_1^3 - u_1^2 u_2 \end{cases} \qquad \text{(c)} \begin{cases} u_1' = 2u_1 u_2 + u_1^3 \\ u_2' = -u_1^2 + u_2^5 \end{cases}$$

(5) Use Poincare-Bendixson theorem to prove the existence of a nontrivial periodic solution of the system

$$\begin{cases} u_1' = u_2 \\ u_2' = -u_1 + (1 - 2u_1^2 - 3u_2^2) u_2. \end{cases}$$

(6) Let φ, ψ, χ be real-valued continuous functions on an interval $[a, b]$. Let $\chi(t) > 0$ on an interval $a \le t \le b$ and suppose that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s)\, ds, \qquad \forall\, t \in [a, b].$$

Prove that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s) \exp\left(\int_s^t \chi(u)\, du \right) ds, \qquad \forall\, t \in [a, b].$$

國立政治大學應用數學系九十三學年度第一學期研究生學科考試試題

科目：微分方程式

(1) Show that the differential equation

$$\frac{d^2y}{dt^2} + (1+t)y = 0$$

is oscillatory in $0 \le t < \infty$.

(2) Compute the first few Picards's iterates with $y_0(x) \equiv 0$ for the initial value problem

$$\frac{dy}{dx} = xy + 2x - x^3,\ y(0) = 0$$

and show that they converge to the solution $y(x) = x^2$ for all x.

(3) For each of the following systems construct a Lyapunov function of the form $c_1u_1^2 + c_2u_2^2$ to determine whether the trivial solution is stable, asymptotically stable or unstable.

$$\text{(a)} \begin{cases} u_1' = -u_1 + e^{u_1}u_2 \\ u_2' = -e^{u_1}u_1 - u_2 \end{cases} \qquad \text{(b)} \begin{cases} u_1' = -u_1^3 + u_1^2u_2 \\ u_2' = -u_1^3 - u_1^2u_2 \end{cases} \qquad \text{(c)} \begin{cases} u_1' = 2u_1u_2 + u_1^3 \\ u_2' = -u_1^2 + u_2^5 \end{cases}$$

(4) Show that the initial value problem

$$\begin{cases} \dfrac{dy}{dx} = 2xy^2 \\ y(0) = 1 \end{cases}$$

has a unique solution and this solution exists only in the interval $-1 < x < 1$.

(5) Let φ, ψ, χ be real-valued continuous functions on an interval $[a, b]$. Let $\chi(t) > 0$ on an interval $a \le t \le b$ and suppose that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s)\, ds,\ \forall t \in [a, b].$$

Prove that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s) \exp\left(\int_s^t \chi(u)\, du\right)\, ds,\ \forall t \in [a, b].$$

(6) Let ϕ and ψ be the solutions of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

in the interval $a \le x \le b$. Suppose that $\phi(x_0) = \psi(x_0) = 0$ for some $a \le x_0 \le b$. Show that

$$\phi(x) = C\psi(x),\ \forall a \le x \le b,$$

for some constant C.

國立政治大學應用數學系九十三學年度第二學期研究生學科考試試題

科目：微分方程式

(1) Suppose that $f = f(x,y)$ is a continuous function in some domain $D \subseteq \mathbb{R}^2$ and satisfies the Lipschitz condition with respect to y in D. Suppose also that $(x_0, y_0) \in D$. Prove that the following initial value problem

$$\begin{aligned} \frac{dy}{dx} &= f(x,y), \\ y(x_0) &= y_0, \end{aligned}$$

has a unique solution defined on the interval $|x - x_0| \leq h$, for some $h > 0$.

(2) Use Poincare-Bendixson theorem to prove the existence of a nontrivial periodic solution of the system

$$\begin{cases} u_1' = 2u_1 - 2u_2 - u_1(u_1^2 + u_2^2) \\ u_2' = 2u_1 + 2u_2 - u_2(u_1^2 + u_2^2). \end{cases}$$

(3) Determine the type of stability of the critical point $(0,0)$ of the following system and sketch the phase portrait

$$\begin{aligned} u_1' &= -2u_1 + u_2 \\ u_2' &= -5u_1 - 6u_2. \end{aligned}$$

(4) Compute the first three Picard's iterates with the initial approximation $y_0(x) \equiv x$ for the initial value problem

$$y' = x^2 - y^2 - 1, \ y(0) = 0.$$

(5) Suppose that a_{11}, a_{12}, a_{21}, a_{22} are real continuous functions on $[a,b]$. Let

$$\begin{aligned} x &= f_1(t), \\ y &= g_1(t), \end{aligned} \quad \text{and} \quad \begin{aligned} x &= f_2(t), \\ y &= g_2(t), \end{aligned}$$

be two solutions of the system

$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y, \ \frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y,$$

on $[a,b]$, and let $W(t)$ be their Wronskian determinant. Let t_0 be a number in $[a,b]$.

(a) Show that $W(t) = W(t_0)\exp(\int_{t_0}^{t} a_{11}(s) + a_{22}(s)\, ds)$, $a \leq t \leq b$.

(b) Show that either $W(t) = 0$ for all $t \in [a,b]$ or $W(t) = 0$ for no $t \in [a,b]$.

(6) For each of the following systems construct a Lyapunov function of the form $c_1u_1^2 + c_2u_2^2$ to determine whether the trivial solution is stable, asymptotically stable or unstable.

$$\text{(a)} \begin{cases} u_1' = -u_1 + e^{u_1}u_2 \\ u_2' = -e^{u_1}u_1 - u_2 \end{cases} \qquad \text{(b)} \begin{cases} u_1' = -u_1^3 + u_1^2u_2 \\ u_2' = -u_1^3 - u_1^2u_2 \end{cases} \qquad \text{(c)} \begin{cases} u_1' = u_1^3 - u_2 \\ u_2' = u_1 + u_2^3 \end{cases}$$

國立政治大學應用數學系九十三學年度第二學期研究生學科考試試題

科目：微分方程式

1. State and prove the existence theorem of Picard-Lindelof.

2. State and prove the Sturm's comparison theorem.

3. Let the matrix A and the vector b be integrable functions of t over $[a,b]$. Let $|A(t)| \leq k(t)$ and $|b(t)| \leq k(t)$, where $\int_a^b k(t)\,dt < \infty$. Let $\tau \in [a,b]$ and consider the initial-value problem $x' = A(t)x + b(t)$, $x(\tau) = \xi$. Use successive approximation method to prove that there is a unique solution φ over $[a,b]$ in the sense that $\varphi \in C[a,b]$ and $\varphi(t) = \xi + \int_\tau^t A(s)\varphi(s)\,ds + \int_\tau^t b(s)\,ds$ on $[a,b]$.

4. Construct a Lyapunov function to determine whether the trivial solution of

$$\begin{cases} u_1' = u_1^3 - u_2 \\ u_2' = u_1 + u_2^3 \end{cases}$$

is stable, asymptotically stable or unstable.

5. Let f_1 and f_2 be two solutions of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

on $a \leq x \leq b$, where a_0, a_1, a_2 are continuous functions on $a \leq x \leq b$ and $a_0(x) \neq 0$ for any $a \leq x \leq b$. Prove the Abel's formula: for any $x_0 \in [a,b]$,

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^x \frac{a_1(s)}{a_0(s)}\,ds\right\},\ \forall\ a \leq x \leq b,$$

where $W(x) = W(f_1, f_2)(x)$ denotes the Wronskian of f_1 and f_2.

6. (a) State the Poincare-Bendixson theorem.

   (b) Use Poincare-Bendixson theorem to prove the existence of a nontrivial periodic solution of the system

$$\begin{cases} u_1' = u_2 \\ u_2' = -u_1 + (1 - 2u_1^2 - 3u_2^2)u_2. \end{cases}$$

國立政治大學應用數學系九十四學年度第二學期研究生學科考試試題

科目：微分方程式

1. Let $y(x)$ be a solution of the initial value problem $y' = y - y^2$, $y(0) = y_0$, where $0 < y_0 < 1$. Show that $y_0 < y(x) \le 1$ for all $x > 0$.

2. Solve the initial value problem.

$$u' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u + \begin{pmatrix} x \\ x \end{pmatrix}, \; u(0) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

by using Picard's method of successive approximations.

3. (a) State the Poincare-Bendixson theorem.

   (b) Use (a) to prove the system

$$\begin{aligned} u_1' &= u_2 - \frac{u_1(u_1^2 + u_2^2 - 1)}{\sqrt{u_1^2 + u_2^2}} \\ u_2' &= -u_1 - \frac{u_2(u_1^2 + u_2^2 - 1)}{\sqrt{u_1^2 + u_2^2}} \end{aligned}$$

   has a nontrivial periodic solution.

4. (a) State the Sturm's comparison theorem.

   (b) Use (a) to show that the equation $y'' + (1 + x)y = 0$ is oscillatory in $(0, \infty)$.

5. Determine the type of stability of the critical point $(0, 0)$ of the system

$$\begin{aligned} u_1' &= 7u_1 + u_2 \\ u_2' &= -3u_1 + 4u_2 \end{aligned}$$

and sketch the phase portraits.

6. Find the adjoint equation to the equation $x^2y'' + 3xy' + 3y = 0$.

國立政治大學應用數學系九十五學年度第一學期研究生學科考試試題

科目：微分方程式

1. Consider the differential equation

$$y^{(4)} + 4y^{(3)} + 5y'' + 4y' + 4y = 0$$

   (a) Find the general solution;

   (b) Find the initial condition on $y(0)$, $y'(0)$, $y''(0)$, and $y^{(3)}(0)$ so that there is a solution $y(t)$ such that (i) $y(t)$ is periodic; (ii) $\lim_{t\to\infty} |y(t)| = 0$; (iii) $y(t)$ is bounded.

2. Let $x' = -\text{grad}V(x)$ be a gradient system, where $V : U \to \mathbb{R}$ is a C^2 function and $U \subseteq \mathbb{R}^n$ is open. Show that $\dot{V}(x) \leq 0$ for all $x \in U$; and $\dot{V}(x) = 0$ if and only if x is an equilibrium of the gradient system.

3. Let A be an operator on $\mathbb{R}^n$. Then the initial value problem

$$\begin{cases} x' = Ax \\ x(0) = k \in \mathbb{R}^n \end{cases}$$

   has the solution of the form $e^{tA}k$ and there are no other solutions.

4. Find the general solution of the system

$$\begin{cases} x_1' = -x_2, \\ x_2' = x_1 + t. \end{cases}$$

5. Solve the boundary value problem:

$$\begin{cases} xy'' - y' - 4x^3 y = 0, \\ y(1) = y(2) = 0. \end{cases}$$

6. Let $u(x)$, $p(x)$, and $q(x)$ be nonnegative continuous functions in $|x - x_0| \leq a$ and

$$u(x) \leq p(x) + \left| \int_{x_0}^{x} q(t)u(t)\,dt \right|, \; |x - x_0| \leq a.$$

   Then we have

$$u(x) \leq p(x) + \left| \int_{x_0}^{x} p(t)q(t) \exp\left(\left| \int_{t}^{x} q(s)\,ds \right| \right) dt \right|$$

   for $|x - x_0| \leq a$.

國立政治大學應用數學系九十五學年度第二學期研究生學科考試試題

科目：微分方程式

1. State and prove an existence and uniqueness theorem.

2. (a) State the Poincare-Bendixon theory.

   (b) Use Poincare-Bendixon theorem to prove the existence of a limit cycle for the system

$$\begin{cases} \frac{dx}{dt} = 4x - 4y - x(x^2 + y^2) \\ \frac{dy}{dt} = 4x + 4y - y(x^2 + y^2). \end{cases}$$

3. Let $f : R \to R$ a bounded and continuous function. For each $n = 1, 2, \ldots$, let $x_n : [0, 1] \to R$ be a solution of the differential equation

$$x' = f(x). \tag{$*$}$$

   If the sequence $\{x_n(0)\}$ converges, prove that there exists a subsequence of $\{x_n\}$ which converges uniformly to a solution of $(*)$ on $[0, 1]$.

4. (a) Show that the differential equation

$$\frac{d^2y}{dt^2} + (1 + t)y = 0$$

   is oscillatory in $0 \le t < \infty$.

   (b) State the theorem used in (a).

5. (a) Give a definition of Lyapunov function.

   (b) Construct a Lyapunov function for the following system to determine whether the trivial solution is stable, asymptotically stable or unstable.

$$\begin{cases} \frac{dx}{dt} = -x + y^2 \\ \frac{dy}{dt} = -y + x^2. \end{cases}$$

   (c) Let φ, ψ, χ be real-valued continuous functions on an interval $[a, b]$. Let $\chi(t) > 0$ on an interval $a \le t \le b$ and suppose that

$$\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s)\, ds, \ \forall\, t \in [a, b].$$

   Prove that $\varphi(t) \le \psi(t) + \int_a^t \chi(s)\varphi(s) \exp(\int_s^t \chi(u)\, du)\, ds, \ \forall\, t \in [a, b]$.

國立政治大學應用數學系九十六學年度第一學期研究生學科考試試題

科目：微分方程式

1. Prove or disprove that the differential equation $y'' + y = f(x)$ has a periodic solution provided that $f(x)$ is periodic.

2. Test the stability, asymptotic stability or unstability for the trivial solution of the following systems:

$$(a)\quad \begin{aligned} u_1' &= \ln(1-u_3) \\ u_2' &= \ln(1-u_1) \\ u_3' &= \ln(1-u_2) \end{aligned} \qquad (b)\ u' = \begin{bmatrix} -1 & e^x \\ 0 & -1 \end{bmatrix} u$$

3. Sketch the phase portraits of the following systems and determine the types of stability of the critical point $(0,0)$.

$$(a)\ u' = \begin{bmatrix} -4 & 2 \\ 0 & -1 \end{bmatrix} u \quad (b)\ u' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} u \quad (c)\ u' = \begin{bmatrix} 1 & 1 \\ -5 & -1 \end{bmatrix} u \quad (d)\ u' = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} u$$

4. (a) State the Sturm's comparison theory.

   (b) Show that every solution of $y'' + (x+1)y = 0$ has an infinite number of positive zeros.

5. State and prove the existence and uniqueness theorem for solution of the initial value problem $y' = f(x,y)$, $y(x_0) = y_0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ a bounded and continuous function. For each $n = 1, 2, \ldots$, let $x_n : [0,1] \to \mathbb{R}$ be a solution of the differential equation

$$x' = f(x). \tag{1}$$

   If the sequence $\{x_n(0)\}$ converges, prove that there exists a subsequence of $\{x_n\}$ which converges uniformly to a solution of (1) on $[0,1]$.