國立政治大學應用數學系九十學年度第二學期研究生學科考試試題

科目：實變函數論

1. Suppose $(X, \mathfrak{M})$ is a measurable space, Y is a topological space, and $f : X \longrightarrow Y$.

   (a) Prove that $\Omega = \{ E \subset Y \mid f^{-1}(E) \in \mathfrak{M} \}$ is a σ-algebra in Y.

   (b) If f is measurable, prove that $f^{-1}(E) \in \mathfrak{M}$ for every Borel set E in Y.

2. Let $x_i > 0$ and $\alpha_i > 0$ for $i = 1, 2, \ldots, n$, where $\alpha_1 + \cdots + \alpha_n = 1$. Prove that

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq \alpha_1 x_1 + \cdots + \alpha_n x_n.$$

3. Let $(X, \mathfrak{M}, \mu)$ be a positive measure space, and suppose $f \in L^1(\mu)$. Prove that for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\int_E |\, f \,|\, d\mu < \epsilon$ where $\mu(E) < \delta$.

4. Does the limit

$$\lim_{k \to \infty} \int_{-1}^{1} e^x \cos\left(\frac{\log k + x}{\sqrt{k} + x} \right) dx$$

   exist? If so, find its value.

5. If $f \in L^1(\mathbb{R})$, and $g \in C^1(\mathbb{R})$ with compact support. Prove that

$$\left| \int_{-\infty}^{\infty} f(x) g^2(x)\, dx \right| \leq \|f\|_{L^1} \int_{-\infty}^{\infty} \left[|\, g(x) \,|^2 + |\, g'(x) \,|^2 \right] dx.$$

6. (a) Evaluate

$$\int_0^{\infty} \int_0^{\infty} \frac{dx\, dy}{(1+y)(1+x^2 y)}.$$

   (b) Deduce from (a) the value of

$$\int_0^{\infty} \frac{\log x}{x^2 - 1}\, dx.$$

7. Let $\{ x_n \}_{n=1}^{\infty}$ be a sequence in a Hilbert space H. We say that $\{ x_n \}$ converges weakly to x in H if for all $y \in H$, $\langle x_n, y \rangle \to \langle x, y \rangle$ as $n \to \infty$.

   (a) Suppose $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ as $n \to \infty$. Prove that

$$\|x_n - x\| \to 0 \text{ as } n \to \infty.$$

   (b) If $\{ x_n \}_{n=1}^{\infty}$ is an orthonormal sequence in H, prove that $x_n \to 0$ weakly as $n \to \infty$.

國立政治大學應用數學系九十一學年度第一學期研究生學科考試試題

科目：實變函數論

1. Show that $L^\infty(0,1)$ is a Banach space, but not separable. (20%)

2. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x > 0, \\ 1, & \text{if } x = 0. \end{cases}$$

Show that the Riemann improper integral $\int_0^\infty f(x)\,dx$ converges, but $f(x)$ is not Lebesgue integrable over $[0,\infty)$. (20%)

3. Prove the Riemann-Lebesgue Lemma: If $f \in L^1(\mathbb{R})$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\cos(nx)\,dx = 0.$$

(20%)

4. State the Fatou's Lemma and give an example where the strict inequality occurs in the Lemma. (20%)

5. Prove the Young's inequality: If $a, b \geq 0$, $1 < p, q < \infty$, and $1/p + 1/q = 1$, then $ab \leq a^p/p + b^q/q$. (20%)

6. State the Radon-Nikodym and Fubini's Theorems. (20%)

國立政治大學應用數學系九十一學年度第二學期學科考試試題

科目：實變函數論

1. Let $C[a,b]$ be the space of all continuous real valued functions defined on the compact interval $[a,b]$.

   (a) Is $C[a,b]$ a closed subspace of $L^\infty([a,b])$?

   (b) Is $C[a,b]$ a closed subspace of $L^1([a,b])$?

   (You should justify your answer!)

2. (a) Let M be a metric space and $\mathfrak{B}$ be a collection of pairwise disjoint open balls in M. Show that if M is separable, then $\mathfrak{B}$ is at most countable.

   (b) Discuss the separability of $L^p(I\!R^n), 1 \le p \le \infty$.

3. State and prove the Riemann-Lebesgue Lemma.

4. Let X be a Banach space and F be a closed subspace of X.

   (a) Define the quotient space X/F.

   (b) Define the quotient norm on X/F.

   (c) Prove, under the quotient norm, X/F becomes a Banach space.

5. Let $E \subseteq I\!R^n$ be a Lebesgue measurable set with finite Lebesgue measure $\lambda_n(E)$. Suppose $f : E \longrightarrow I\!R^*$ is a Lebesgue measurable function and

$$E_k = \{\, x \in E \mid (k-1) \le |\, f(x) \,| < k \,\}, k = 1, 2, 3, \ldots$$

   Show that, for $1 \le p < \infty, f \in L^p(E) \Longleftrightarrow \sum_{k=1}^{\infty} k^p \lambda_n(E_k) < \infty$.

6. Let

$$\beta(x) = \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \ge 1 \end{cases}, x \in I\!R^n$$

   and

$$\alpha(x) = \beta(x) \left(\int_{I\!R^n} \beta(x)\, dx \right)^{-1}, x \in I\!R^n$$

$$\alpha_\epsilon(x) = \epsilon^{-n} \alpha(x/\epsilon), \epsilon > 0, x \in I\!R^n.$$

   Show that

   (a) $\alpha \in C_0^\infty(I\!R^n), supp\alpha = \bar{B}(0;1)$ and $\int_{I\!R^n} \alpha(x)\, dx = 1$.

   (b) $\alpha_\epsilon \in C_0^\infty(I\!R^n), supp\alpha_\epsilon = \bar{B}(0;\epsilon)$ and $\int_{I\!R^n} \alpha_\epsilon(x)\, dx = 1$.

   (c) If $f \in L^p(I\!R^n), 1 \le p < \infty$, then $f * \alpha_\epsilon \longrightarrow f$ in $L^p(I\!R^n)$ as $\epsilon \longrightarrow 0$. In particular, $C_0^\infty(I\!R^n)$ is dense in $L^p(I\!R^n)$.

國立政治大學應用數學系九十二學年度第一學期研究生學科考試試題

科目：實變函數論

1. Let $(X, \mathcal{F}, \mu)$ be a measure space and $\{A_n\}$ be a sequence in $\mathcal{F}$ with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that the set of all points belonging to infinite many A_n's has measure zero.

2. Let $f(x) = e^{-[x]}$, where $[x]$ is the greatest integer function. Is $f(x)$ integrable over $[0, \infty)$ with respect to the Lebesgue measure? If yes, compute its integral.

3. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space, k be a bounded measurable function on X and $1 \leq p \leq \infty$. Define T on $L^p(X, \mathcal{F}, \mu)$ by
$$(Tf)(x) = k(x)f(x), \quad \forall\, x \in X.$$
Show that T is a bounded linear operator on $L^p(X, \mathcal{F}, \mu)$ and compute $\|T\|$ the operator norm of T.

4. Show that the space $C[a, b]$ of continuous functions on $[a, b]$ is not complete with respect to L^2-norm. What is the completion of $C[a, b]$ with respect to L^2-norm.

5. Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\{f_n\}$ be a sequence of measurable functions on X. Discuss the relation of convergence a.e. on X, convergence in measure on X and L^p-convergence of $\{f_n\}$, $1 \leq p \leq \infty$.

6. Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{I}, \nu)$ be two measure spaces and
$$f(x, y) = h(x)g(y), \quad \forall\, (x, y) \in X \times Y,$$
where $h \in L^1(X, \mathcal{F}, \mu)$ and $g \in L^1(Y, \mathcal{I}, \nu)$. Show that f is measurable on the product measurable space $(X \times Y, \mathcal{F} \times \mathcal{I})$ and
$$\int_{X \times Y} f \, d(\mu \times \nu) = \left(\int_X h \, d\mu \right) \left(\int_Y g \, d\nu \right).$$

國立政治大學應用數學系九十二學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let $\|\cdot\|$ and $\|\cdot\|'$ be two norms on a finite-dimensional normed linear space X. Prove that they are equivalent.

2. Let $f \in L^1([0,1])$. Show that $x^n f(x) \in L^1([0,1])$ for all $n = 1, 2, \ldots$, and $\int_0^1 x^n f(x)\,dx \to 0$ as $n \to \infty$.

3. Let (X, S, μ) be a measure space. Identify the dual space $L^2(X, S, \mu)^*$ of $L^2(X, S, \mu)$. You must justify your answer!

4. Let $f \in L^1([a,b])$ and $F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$, be the indefinite integral of f on $[a,b]$. Show that F is continuous on $[a,b]$ and of bounded variation on $[a,b]$.

5. State and prove the Jensen's inequality for integral.

6. Let (X, S) be a measurable space and $\{\mu_n\}$ be a sequence of measures on (X, S) such that $\{\mu_n(E)\}$ is an increasing sequence for all $E \in S$. Show that the set function μ defined by $\mu(E) = \lim_{n\to\infty} \mu_n(E)$, $E \in S$, is a well-defined measure on (X, S).

7. Let X be a compact Hausdorff space and $C(X)$ be the space of real-valued continuous functions on X. Let $\|f\| = \max_{x\in X} |f(x)|$, $f \in C(X)$. Show that $(C(X), \|\cdot\|)$ is a Banach space.

國立政治大學應用數學系九十三學年度第一學期研究生學科考試試題

科目：實變函數論

1. Let (X, β) be a measurable space, $\{\mu_n\}$ a sequence of measures which converge setwise to a measure μ, and $\{f_n\}$ a sequence of nonnegative measurable functions which converge pointwise to the function f. Show that
$$\int f\, d\mu \leq \underline{\lim} \int f_n\, d\mu_n.$$

2. Let (X, β, μ) be a measure space and g a nonnegative measurable function in X. Set
$$\nu E = \int_E g\, d\mu.$$
Prove that ν is a measure in β.

3. If (X, β, μ) be a measure space, $E_i \in \beta$ and $E_i \supset E_{i+1}$, then
$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n.$$

4. Suppose that f is a bounded linear functional in Hilbert space H. Then there exists $y \in H$ such that
$$f(x) =< x, y >,\ \forall x \in H.$$

5. If a linear operator is continuous at one point, it is bounded.

6. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in metric space (X, ρ), then $\rho(x_n, y_n)$ converges.

7. Suppose that $f : [0, 1] \to [0, 1]$ is continuous, then there exists $x \in [0, 1]$ such that
$$f(x) = x.$$

國立政治大學應用數學系九十四學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let (X, A, μ) be a measure space.

   (a) Suppose that $1 \leq p < r < \infty$. Prove that $L^p(X, A, \mu) \cap L^\infty(X, A, \mu) \subseteq L^r(X, A, \mu)$. Moreover, show that if $f \in L^p(X, A, \mu) \cap L^\infty(X, A, \mu)$, then $||f||_r \leq ||f||_p^{p/r} ||f||_\infty^{1-p/r}$.

   (b) If $f \in L^r(X, A, \mu) \cap L^\infty(X, A, \mu)$ for some $1 \leq r < \infty$, then $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

2. Let $\Omega = \{(x, y) \in R^2 \mid 0 < x < \infty, 0 < y < 1\}$ and $f(x, y) = ye^{-xy} \sin x$. Show that

   (a) Ω is a Lebesgue measurable subset of R^2.

   (b) f is a Lebesgue measurable function on Ω.

   (c) Is $f(x, y)$ Lebesgue integrable on Ω? If yes, find the Lebesgue integral of $f(x, y)$ over Ω.

3. Let H be a Hilbert space and $B(H)$ be the collection of all bounded linear operators on H. Show that $B(H)$ is a Banach space with respect to the operator norm of linear operator.

4. For $1 \leq p \leq \infty$, is $L^p(X, A, \mu)$ a Hilbert space? (Justify your answer!)

5. Let λ and μ be σ-finite measures on a measurable space. If $\lambda << \mu$ and $f = d\lambda/d\mu$, then, for all nonnegative measurable functions g on X, we have

$$\int_X g\, d\lambda = \int_X gf\, d\mu.$$

6. Let (X, A, μ) be a measure space and $\{f_n\}$ be a sequence of integrable functions on X such that $f_n \to f$ a.e. on X. Show that if

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0,$$

   then

$$\lim_{n\to\infty} \int_X |f_n|\, d\mu = \int_X |f|\, d\mu.$$

國立政治大學應用數學系九十五學年度第一學期研究生學科考試試題

科目：實變函數論

1. Let $X = C[a,b]$ be the space of continuous functions on $[a,b]$, $||f||_\infty = \max_{a\le t\le b}|f(t)|$ and $||f||_1 = \int_a^b |f(t)|\,dt$, $f \in X$. Show that
   (a) $||\cdot||_\infty$ and $||f||_1$ are norms on X;
   (b) $(X, ||\cdot||_\infty)$ is a Banach space, but $(X, ||\cdot||_1)$ is not;
   (c) $\lim_{p\to\infty}(\int_a^b |f(t)|^p\,dt)^{1/p} = ||f||_\infty$, for all $f \in X$.
2. Let $1 < p < \infty$ and $f \in L^p[0,\infty)$. Show that, for all $x > 0$, $|\int_0^\infty e^{-tx}f(t)\,dt| \le ||f||_p(xq)^{1/q}$ for some $1 < q < \infty$.
3. Let (X, S, μ) be a measure space. What is the dual space of $L^p(X, S, \mu)$, $1 \le p \le \infty$?
4. State and prove the Lebesgue dominated convergence theorem.
5. Let (X, S, μ) be a measure space. Show that $f_n \xrightarrow{\mu} f$ as $n \to \infty$ if and only if
   $$\lim_{n\to\infty}\int_X \frac{|f_n - f|}{1 + |f_n - f|}\,d\mu = 0,$$
   where $\mu(X) < \infty$.
6. Describe the way of constructing the Lebesgue measure λ_n on $\mathbb{R}^n$.

國立政治大學應用數學系九十五學年度第二學期研究生學科考試試題

科目：實變函數論

1. Let (X, S, μ) be a measure space and f be a nonnegative integrable function on X. Show that the set function ν defined by
$$\nu(E) = \int_E f \, d\mu, \ E \in S,$$
is a finite measure on (X, S) and $\nu << \mu$.

2. State and prove the Lebesgue dominated convergence theorem.

3. Let ν be a signed measure on a measurable space (X, S). Show that there exists a unique pair of measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

4. Let M be a compact metric space. Show that

   (a) Every continuous function $f : M \to R$ admits both maximum and minimum values.

   (b) Every closed subset of M is compact.

5. (a) What is complete measure space?

   (b) Show that every measure space admits a completion.

6. Let (X, S, μ) be a finite measure space and $f \in L^\infty(X, S, \mu)$.

   (a) Show that $f \in L^p(X, S, \mu)$ for all $1 \leq p < \infty$ and
$$\lim_{p \to \infty} ||f||_p = ||f||_\infty.$$

   (b) Evaluate
$$\lim_{p \to \infty} \left(\int_0^1 e^{-px^2} \, dx \right)^{1/p}$$
   if exists.

國立政治大學應用數學系九十六學年度第一學期研究生學科考試試題

科目：實變函數論

1. Define $(Tf)(x) = \frac{1}{x}\int_0^x f(y)\,dy$ and $(Sf)(x) = \int_x^\infty \frac{f(y)}{y}\,dy$. Show that T and S are bounded linear operators on $L^p(0,\infty)$ and $L^q(0,\infty)$, respectively, where $1 < p \leq \infty$ and $1 \leq q < \infty$. Moreover, $||T|| \leq \frac{p}{p-1}$ if $1 < p < \infty$ and $||T|| \leq 1$ if $p = \infty$; $||S|| \leq q$.

2. Let Y be a topological space and X be a set. If $\{f_\alpha\}_{\alpha \in I}$ is a collection of mappings from X into Y, can you define a topology on X so that each f_α, $\alpha \in I$, becomes a continuous mapping from X into Y?

3. Let X be a normed linear space and N be a closed subspace of X. Describe the quotient space and the quotient norm on X/N; Moreover, if X is a Banach space, then so is X/N under quotient norm.

4. Let (X, S, μ) be a measure space, $1 \leq p < \infty$ and $f \in L^p(X, S, \mu)$. Show that the set $\{x \in X \mid f(x) \neq 0\}$ is of σ-finite and $\mu(\{x \in X \mid |f(x)| \geq n\}) \to 0$ as $n \to \infty$.

5. Prove the following statements:

   (a) Let $f_n = n^{-1/p}\chi_{[0,n]}$, $n = 1, 2, \ldots$, where $1 \leq p < \infty$. Show that $f_n \to 0$ uniformly on $\mathbb{R}$, but $f_n \nrightarrow 0$ in $L^p(\mathbb{R})$.

   (b) Let $g_n = \chi_{[n,n+1]}$, $n = 1, 2, \ldots$. Show that $g_n \to 0$ pointwise on $\mathbb{R}$, but $g_n \nrightarrow 0$ in measure.

6. Let (X, S) be a measurable space and ν be a signed measure on (X, S). Show that

   (a) If $\{A_n\}$ is a sequence of positive set w.r.t. ν, then so is $\bigcup_{n=1}^\infty A_n$.

   (b) There exists a partition $\{A, B\}$ of X with A positive set and B negative set w.r.t. ν.

   (c) Describe the positive, negative and total variation of ν.