國立政治大學 應用數學系 106 學年度第一學期　學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	實變函數論	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 下午 14:00~17:00	試題編號 Course	

本試卷共有 6 個題目，
碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。
博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Show that every nonempty open set S in $\mathbb{R}$ is the union of a countable collection of disjoint component intervals of S and characterize all open connected subsets of $\mathbb{R}$. (Justify your answer)

2. (a) Let $\overline{B}(x,r)$ be a closed ball in $\mathbb{R}^n$ and $f:\overline{B}(x,r)\to\overline{B}(x,r)$ be a mapping satisfying $\|f(x')-f(x'')\|\le\frac{1}{2}\|x'-x''\|$ for all $x',x''\in\overline{B}(x,r)$. Show that f has a unique fixed point; (b) show that all norms on $\mathbb{R}^n$ are equivalent.

3. (a) Let $f\in L^1(\mathbb{R})$. Show that the function $F(y)=\int_{\mathbb{R}} f(x+y)\,dm(x)$, $y\in\mathbb{R}$, is a well-defined continuous function on $\mathbb{R}$.
(b) In the space $C[0,1]$ of all continuous real-valued functions on $[0,1]$. Define, for $f\in C[0,1]$, $\|f\|_1=\int_0^1|f(x)|\,dx$. Show that $\|\cdot\|_1$ is a norm on $C[0,1]$. Is it a Banach space?

4. (a) Show that the function $\varphi(x)=-\log x$ is convex on $(0,\infty)$.
(b) Using Jensen's inequality to show that $\prod_{i=1}^{n}x_i^{\alpha_i}\le\sum_{i=1}^{n}\alpha_i x_i$, where $x_i\ge 0$, $\alpha_i>0$, $1\le i\le n$, and $\sum_{i=1}^{n}\alpha_i=1$.

5. State and prove the Lebesgue dominated convergence theorem.

6. Let $\{x_n\}$ be a real sequence and $\sum_{n=1}^{\infty}x_n^4$ converges absolutely. Does $\sum_{n=1}^{\infty}\frac{x_n}{n}$ converges absolutely? (Justify your answer).

本考試：☑ 不需使用簡易計算機，☐ 使用簡易計算機　←請出題老師勾選，謝謝！

命題老師：(Teacher)	(簽章) 106 年 9 月 9 日 (Signature & date)	試題隨卷繳交

命題紙使用說明：試題將用原件印製，敬請使用黑色墨水正楷書寫或打字（紅色不能製版請勿使用）。

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國立政治大學 應用數學系 106 學年度第一學期 學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	微分方程式	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

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碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。

博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Sketch the phase portraits of the following systems and determine the types of stability of equilibrium point (i.e. critical point or fixed point) $(0,0)$.

(i) $u' = \begin{bmatrix} -2 & 1 \\ -5 & -6 \end{bmatrix} u$ (ii) $u' = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix} u$ (iii) $u' = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} u$ (iv) $u' = \begin{bmatrix} -2 & -5 \\ 2 & 2 \end{bmatrix} u$

2. Consider the nonlinear differential system

$$\begin{cases} x' = 1 - xy \\ y' = x - y^3. \end{cases}$$

Find all equilibrium points of the system, and discuss their stability.

3. For each of the following systems, construct a Lyapunov function of the form $c_1x^2 + c_2y^2$ to determine whether the trivial solution is "stable", "asymptotically stable", or "unstable":

(i) $\begin{cases} x' = -x^3 + x^2y \\ y' = -x^3 - x^2y \end{cases}$ (ii) $\begin{cases} x' = -x + 2x^2 + y^2 \\ y' = -y + xy \end{cases}$

4. Prove the existence of a nontrivial periodic solution of the following differential system.

$$\begin{cases} x' = 2x - 2y - x(x^2 + y^2) \\ y' = 2x + 2y - y(x^2 + y^2) \end{cases}$$

5. Let $p(t)$ and $q(t)$ be continuous functions in $[t_0, \infty)$. Suppose that all solutions of $x'' + p(t)x = 0$ are bounded in $[t_0, \infty)$. Show that all solutions of $x'' + (p(t) + q(t))x = 0$ are bounded in $[t_0, \infty)$ provided that $\int_{t_0}^{\infty} |q(t)|dt < \infty$.

6. Prove or disprove the following statement.

The solution to the equation

$$u'' + u^{\frac{1}{3}} = 0, u(0) = u'(0) = 0,$$

is unique.

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命題老師：(Teacher)	（簽章）106 年 9 月 4 日 (Signature & date)	試題隨卷繳交

國立政治大學 應用數學系 106 學年度第一學期 學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	組合學	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

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博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Show that $K_{3,3}$ is nonplanar.

2. Show that $\sum_{k=0}^{m} C_k^m C_{r+k}^n = C_{m+r}^{m+n}$.

3. Suppose that $2n$ points are arranged on the circumference of a circle. Pair up these points and join corresponding points by chords of the circle. How many ways to do this pairing so that none of the chords cross?

4. Find the number of ways to arrange flags on the n-foot flagpole using three types of flags: red flags 2 feet high, yellow flags 1 foot high, and blue flags 1 foot high.

5. Given $A_1, A_2, \cdots, A_n \subseteq U$ and $\mathrm{I} \subseteq \{1, 2, \cdots, n\}$.
Show that the number of elements in exactly m sets is
$\sum_{k=0}^{n-m} (-1)^k C_m^{m+k} \sum_{|I|=m+k} |\cap_{i \in I} A_i|$.

6. How many ways are there to print six edges of a tetrahedron using m colors?

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命題老師：(Teacher)	簽章 (Signature & date) 106 年 9 月 10 日	試題隨卷繳交

國立政治大學 應用數學系 106 學年度第一學期　學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM　　Page　/Total

考試科目 Course	作業研究	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

本試卷共有 6 個題目，
碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。
博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

I. Consider the following problem.

$$\begin{array}{lrrrrl} \max Z & = & 5x_1 & +2x_2 & +3x_3 & \\ \text{subject to} & & x_1 & +5x_2 & +2x_3 & \leq b_1 \\ & & x_1 & -5x_2 & -6x_3 & \leq b_2 \\ & & x_1, & x_2, & x_3 \geq 0 & \end{array}$$

The following optimal tableau corresponds to specific values of b_1 and b_2:

Basic	x_1	x_2	x_3	x_4	x_5	Solution
Z	0	a	7	d	e	150
x_1	1	b	2	1	0	30
x_5	0	c	-8	-1	1	10

Determine the following and write down the calculation procedures to get the values:

(a) The right-hand-side value b_1 and b_2.

(b) The elements a, b, c, d, e.

(c) The optimal dual solution of this problem.

II. Show by duality that if the problem

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b \qquad (1) \\ & x \geq 0 \\ & c, x \in R^n, A \in R^{m\times n}, b \in R^m \end{aligned}$$

has a finite optimal solution, then the new problem with the same A and c

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b' \qquad (2) \\ & x \geq 0 \\ & c, x \in R^n, A \in R^{m\times n}, b' \in R^m \end{aligned}$$

can not be unbounded below, no matter what value the vector b' might take.

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命題老師：(Teacher)　（簽章）(Signature & date) 106 年 9 月 6 日　試題隨卷繳交

國立政治大學 應用數學系 106 學年度第一學期　學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM　Page /Total

考試科目 Course	作業研究	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

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博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

III. A company sells an item whose demand over the next 4 months is 100, 140, 210, and 180 units, respectively. The company can stock just enough supply to meet each month's demand, or it can overstock to meet the demand for two or more successive and consecutive months. In the latter case, a holding cost of \$1.2 is charged per overstocked unit per month. The company estimates the unit purchase prices for the next 4 months to be \$13, \$15, \$10, and \$12, respectively. A setup cost of \$200 is incurred each time a purchase order is placed. The company wants to develop a purchasing plan that will minimize the total costs of ordering, purchasing, and holding the item in stock.

(a) Define the following variables:

x_j = amount of purchase in month j, $j = 1, 2, 3, 4$.

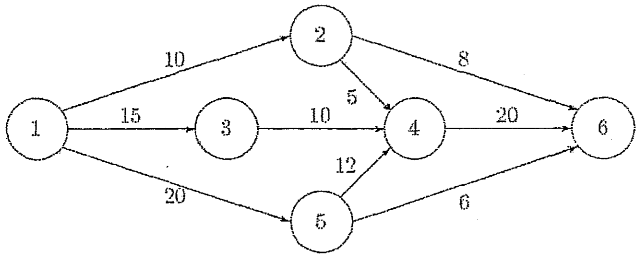
y_j = 1 if there is nonzero purchase in month j; 0 otherwise.

z_j = inventory remaining at the end of month j.

Give a Mixed Integer Linear Programming formulation for the problem.

(b) Formulate the problem as a shortest-route model and solve it.

IV. Consider the following network where ① is the source node, ⑥ is the sink node, and the numbers along the arcs denote the capacities of flows:



(a) Illustrate a cut separating the source and sink and give the capacity of your cut.

(b) Find the maximum flow from the source to the sink using augmenting path algorithm. Identify the associated minimum cut of your solution, and verify the max-flow min-cut theorem.

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命題老師：(Teacher)	（簽章）(Signature & date) 106年 9月 6日	試題隨卷繳交

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國立政治大學 應用數學系 106 學年度第一學期 學科 考試試題

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V. Suppose that the one-step transition probability matrix of a Markov chain is given as follows:

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0.6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) For each absorbing state, find the probability of absorption into that state.

(b) For each transient state, find the mean amount of time spent occupying that state.

(c) The mean amount of time spent before absorption.

VI. Consider a self-service model in which the customer is also the server. Note that this corresponds to having an infinite number of servers available. Customers arrive according to a Poisson process with parameter λ, and service times have an exponential distribution with parameter μ.

(a) Construct the rate diagram for this queueing system. Let P_n denote the stationary probability of n customers in the system. Use the balance equations to find the expression for P_n in terms of P_0. Find P_0.

(b) Find the average customers in the system L and the average waiting time in the system W.

(c) Find the average queue length L_q and the average waiting time in queue W_q.

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命題老師：(Teacher)	（簽章）106年 9月 6日 (Signature & date)	試題隨卷繳交

（ 榮譽第一 ）

國立政治大學 應用數學系 106 學年度第一學期 學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	數理統計	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

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1. Let (X, Y) be a random point chosen uniformly on the region $R = \{(x,y) \mid |x| + |y| \le 2\}$.
   (1) Find the conditional density of X, given $Y = -1$.
   (2) Find $Var(X \mid Y = -1)$.

2. Let (X, Y) have joint density $f(x,y) = cxy$, $0 < x < y < 1$.
   (1) Find c and $P(X < \frac{1}{2})$.
   (2) Find the density of $U = X/Y$.

3. Let $X_1, \cdots, X_n \overset{iid}{\sim} f(x;\theta) = \theta x^{\theta - 1}$, $0 < x < 1$.
   (1) Find the lower bound for an unbiased estimator of θ.
   (2) Find the best unbiased estimator of θ.

4. (1) Let X_1, X_2, X_3 be independent, with $X_i \sim P(i\theta)$. Find the UMP size -0.05 test that $\theta = 1$ against $\theta < 1$.
   (2) Let $X_1, \cdots, X_n$ be independent, with $X_i \sim N(\theta, 1)$. Find the size $-\alpha$ LRT for testing that $\theta = 2$

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國立政治大學 應用數學系 106 學年度第一學期 學科 考試試題
NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	數理統計	開課系級 Dept, & Class	研究所	日期 Date, Period	106 年 9 月 18 日 上午 9:00~12:00	試題編號 Course No.	

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博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

against $\theta \neq 2$.

5. (1) Let $Y \sim E(\lambda)$, and $X|Y$ have a Poisson distribution with mean Y. Find the marginal density of X.

(2) Let $X = (X_1, X_2, X_3, X_4) \sim M_4\left(18, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{6}{12}\right)$. Find $P(X_2 = 3 \mid X_1 = 2)$.

6. (1) State the Rao-Blackwell theorem.

(2) Let $X_1, \cdots, X_n$ be independent, with $X_i \sim E(i\theta)$, $\theta > 0$. Find a $1-\alpha$ confidence interval for θ.

■ $X \sim N(\mu, \sigma^2) \iff f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $-\infty < x < \infty$

■ $X \sim P(\lambda) \iff f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \cdots$

■ $X \sim E(\theta) \iff f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x > 0$.

■ $(X_1, X_2, X_3, X_4) \sim M_4(n, p_1, p_2, p_3, p_4) \iff f(x_1, x_2, x_3, x_4) = \frac{n!}{x_1!\,x_2!\,x_3!\,x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$, $x_1 + x_2 + x_3 + x_4 = n$, $x_i \geq 0$.

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命題老師：(Teacher)	（簽章）(Signature & date) 106 年 9 月 9 日	試題隨卷繳交

TABLE 6 POISSON DISTRIBUTION FUNCTIONS. $P(X \leq x), X \sim P(m)$.

x \ m	.10	.20	.30	.40	.50	.60	.70	.80	.90
0	.905	.819	.741	.670	.607	.549	.497	.449	.407
1	.995	.982	.963	.938	.910	.878	.844	.809	.772
2	1.000	.999	.996	.992	.986	.977	.966	.953	.937
3		1.000	1.000	.999	.998	.997	.994	.991	.987
4				1.000	1.000	1.000	.999	.999	.998
5							1.000	1.000	1.000

x \ m	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	15.0
0	.368	.135	.050	.018	.007	.002	.001	.000	.000		
1	.736	.406	.199	.092	.040	.017	.007	.003	.001		
2	.920	.677	.423	.238	.125	.062	.030	.014	.006	.003	
3	.981	.857	.647	.433	.265	.151	.082	.042	.021	.010	
4	.996	.947	.815	.629	.440	.285	.173	.100	.055	.029	.001
5	.999	.983	.916	.785	.616	.446	.301	.191	.116	.067	.003
6	1.000	.995	.966	.889	.762	.606	.450	.313	.207	.130	.008
7		.999	.988	.949	.867	.744	.599	.453	.324	.220	.018
8		1.000	.996	.979	.932	.847	.729	.593	.456	.333	.037
9			.999	.992	.968	.916	.830	.717	.587	.458	.070
10			1.000	.997	.986	.957	.901	.816	.706	.583	.118
11				.999	.995	.980	.947	.888	.803	.697	.185
12				1.000	.998	.991	.973	.936	.876	.792	.268
13					.999	.996	.987	.966	.926	.864	.363
14					1.000	.999	.994	.983	.959	.917	.466
15						.999	.998	.992	.978	.951	.568
16						1.000	.999	.996	.989	.973	.664
17							1.000	.998	.995	.986	.749
18								.999	.998	.993	.819
19								1.000	.999	.997	.875
20									1.000	.998	.917
21										.999	.947
22										1.000	.967
23											.981
24											.989
25											.994
26											.997
27											.998
28											.999
29											1.000