(榮譽第一)

國立政治大學 應用數學系 100 學年度第 一 學期 學科 考試試題

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

考試科目 Course	實變函數論	開課系級 Dept, & Class	研究所	日期 Date, Period	100 年 9 月 19 日 上午 9:00~12:00	試題編號 Course No.	

本試卷共有 6 個題目，
碩士班：請選 5 題作答，每題 20 分，請在答案卷最前面註明所選的 5 題，否則依學生作答之前 5 題計分。
博士班：6 題全做答，每題 17 分，超過 100 分則以 100 分計。

1. Let

$$f(x)=\begin{cases}\frac{\sin x}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$$

(a) Does the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exist as an improper Riemann integral?

(b) Is $f(x)$ Lebesgue integrable over $(-\infty,\infty)$?

Prove your answer!

2. Let $p(x)$ be a polynomial function given by $p(x)=\sum_{i=1}^{k} a_i x^i$.

(a) Show that for any positive y and natural number n, $|p(y/n)|\leq \sum_{i=1}^{k} |a_i| y^i$.

(b) Show that

$$n\int_0^{\infty} p(x)e^{-nx}dx \to p(0) \quad \text{as} \quad n\to\infty.$$

3. Let X be a normed linear space. Show that the set X^* of all bounded linear functionals on X is a Banach space.

4. Let $f\in L^1(\mathbb{R})$ be a uniformly continuous function on $\mathbb{R}$. Show that $\lim_{|x|\to\infty} f(x)=0$.

5. Suppose that $\{f_n\}$ is a sequence of nonnegative integrable functions such that

$f_n\to f$ a.e., with f integrable, and $\int_{\mathbb{R}} f_n \to \int_{\mathbb{R}} f$. Prove that $\int_{\mathbb{R}} |f_n - f| \to 0$.

6. Suppose that $f\in L^1(\mathbb{R})$ is a absolutely continuous function on $\mathbb{R}$. Show that if in addition

$$\lim_{t\to 0+}\int_{\mathbb{R}}\left|\frac{f(x+t)-f(x)}{t}\right| dx = 0$$

Then $f\equiv 0$.

本考試：☑不需使用簡易計算機，□ 使用簡易計算機 ←請出題老師勾選，謝謝！

考試科目 Course	數理統計	開課系級 Dept, & Class	研究所	日期 Date, Period	100 年 9 月 19 日 上午 9:00~12:00	試題編號 Course No.	

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To earn your credits, you must show your work.

You don't have to use the calculator. However, you may use the information at the end of page 2 and the attached statistical table.

1. Let X_1, X_2, X_3 be independent with X_i having density $f(x_i) = \exp(-x_i)$, $x_i>0$. Let $U_1=X_1+X_2+X_3$, $U_2=X_2/U_1$, and $U_3=X_3/U_1$. (a)Find the joint density of U_1, U_2, and U_3. (b) Find the marginal density of U_1. (c) Find the conditional density of U_1 given U_2 and U_3.

2. A prisoner is in a cell with four doors. He chooses a door at random (each with probability 1/4). The first door leads to a tunnel which leads to freedom in one day. The second door leads to a long tunnel which leads to freedom in three days. The third tunnel is a trap which leads back to the cell in two days, and the forth tunnel is also a trap which leads back to the prison cell, but in five days. Each time the prisoner gets back to the cell, he chooses a door at random from the four doors (again, each with probability 1/4). (That is, he does not remember the door he chose the previous time.) Find the expected time till the prisoner escapes.

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3. Suppose we observe X_i , i=1, …, n, independent, with $X_i \sim E(i\theta)$, $\theta > 0$, that is, $f(x_i;\theta) = (i\theta)^{-1} \exp(\frac{-x_i}{i\theta})$, θ>0. (a) Find the MLE $T_n(X_1, X_2, \ldots, X_n)$ of θ. (b) Find the asymptotical distribution of T_n as n goes to infinity.

4. State and prove the Neyman-Pearson Theorem.

5. Let $X_1, X_2, \ldots, X_n$ be independent, and let X_i be normal distribution with mean iθ and variance 1. Find the uniformly most powerful (UMP) size-0.025 test that θ =3 against θ < 3 when n=3.

6. Let X_1 and X_2 be two independent and identical Bernoulli distributions with mean θ. That is, $X_i \sim B(1, \theta)$ for i=1 and 2. Consider testing θ=0.5 against θ>0.5. Let $\Phi(X_1, X_2)$ be the nonrandomized test which rejects the null hypothesis if X_1=1. (a) Show that $T=X_1+X_2$ is a sufficient statistic. (b) Find $\Phi^*(T)=E\langle\Phi(X_1,X_2)|T\rangle$. (c) Which test ($\Phi(X_1,X_2)$ or $\Phi^*(T)$) has more power? Why?

Note:

x	6	7	8	10	11	12	13	14	15
$\sqrt{x}$	2.45	2.65	2.83	3.16	3.32	3.46	3.61	3.74	3.87

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考試科目 Course	組合學	開課系級 Dept, & Class	研究所	日期 Date, Period	100 年 09 月 19 日 上午 09:00~12:00	試題編號 Course No.	

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★ Show *all* your work for credits!

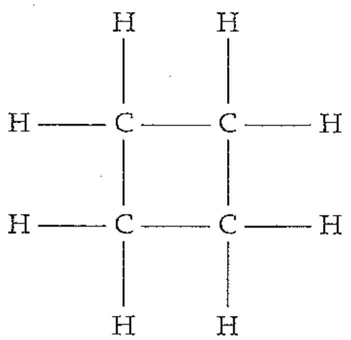
1. Suppose α is an irrational real number. Then there are infinitely many rational numbers p/q such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

2. We say that a rooted tree is *strictly binary* if every parent vertex has exactly two children. How many strictly binary trees are there with k parent vertices? (Do not take symmetry into account: If two trees are mirror images of one another, count both configurations.)

3. If $2 \le p' \le p$ and $2 \le q' \le q$, then prove that $R(p',q') \le R(p,q)$ where $R(\cdot,\cdot)$ is the Ramsey number associated with integers. Also, prove that equality holds if and only if $p' = p$ and $q' = q$.

4. A cyclobutane is a hydrocarbon constructed of 4 carbon atoms arranged cyclically with 2 hydrogen atoms attached to each carbon, as illustrated in the following:



   (a) How many isomers can be obtained by replacing 2 hydrogens with nitrogen and 3 with oxygen?
   (b) Find the number of isomers with 3 hydrogens.

5. Let $n > 1$ be an integer. A *conference matrix* M of order n is an $n \times n$ matrix with 0's on the diagonal and +1 or −1 in all other positions, and with the property

$$MM^t = (n-1)I_n$$

where I_n is the identity matrix of order n and M^t is the transpose of M.

   (a) Show that n must be even.
   (b) Show that permuting rows and columns and multiplying rows and columns by −1, we can obtain a matrix that is symmetric if $n \equiv 2 \bmod 4$ and antisymmetric if $n \equiv 0 \bmod 4$.

6. Let $GF(4) = \{0, 1, \omega, \overline{\omega}\}$, where $\omega^3 = 1$ and $\overline{\omega} = \omega^2$.

   (a) Give a parity-check matrix for the Hamming single error-correcting code C of length 5 over $GF(4)$.
   (b) Give a generator matrix for C.
   (c) What is the minimum weight of C? Give reasons.

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In the Problem 4, you must graphically visualize the cyclobutane in the 3 dimensional space.

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考試科目 Course	微分方程式	開課系級 Dept, & Class	研究所	日期 Date, Period	100 年 9 月 19 日 上午 9:00~12:00	試題編號 Course No.	

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1. Prove Gronwall's Lemma: Suppose that $g(t)$ is a nonnegative continuous function and

$$g(t) \le C + K\int_0^t g(s)ds$$

for all $t \in [0,a]$, where C and K are positive constants. Show that $g(t) \le Ce^{Kt}$, for all $t \in [0,a]$.

2. Prove that the initial-value problem

$$\frac{dy}{dt} = e^{\sin t}y, \qquad y(0) = y_0$$

with y_0 given, has a unique solution on $[0,\infty)$.

3. Show that the system

$$\frac{dx}{dt} = x\left(\lambda - \left(x^2 + (1+\epsilon^2)y^2\right)\right) + \omega y,$$

$$\frac{dy}{dt} = y\left(\lambda - \left(x^2 + (1+\epsilon^2)y^2\right)\right) - \omega x$$

has a limit cycle for $\lambda, \epsilon > 0$.

4. Construct a Lyapunov function to determine whether the trivial solution

$$x_1' = -x_1 + 2x_2^2$$
$$x_2' = -x_2 + x_1^2$$

is stable, asymptotically stable or unstable.

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5. Let $a(t)$ and $b(t)$ be continuous and periodic of period ω. Let ϕ_1 and ϕ_2 be solutions of $x'' + a(t)x' + b(t)x = 0$ such that $\phi_1(0) = 1$, $\phi_1'(0) = 0$, $\phi_2(0) = 0$ and $\phi_2'(0) = 1$. Show that the multipliers are solutions of $\lambda^2 - A\lambda + B = 0$, where $A = \phi_1(\omega) + \phi_2'(\omega)$ and $B = \exp\left[-\int_0^\omega a(t)\,dt\right]$.

6. Determine the type of stability of the critical point $(0,0)$ of each of the following systems and sketch the phase portraits.

(a) $\begin{cases} \dfrac{dx}{dt} = 7x + y \\ \dfrac{dy}{dt} = -3x + 4y \end{cases}$ (b) $\begin{cases} \dfrac{dx}{dt} = -2x - 5y \\ \dfrac{dy}{dt} = 2x + y \end{cases}$

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