國立政治大學 **112** 學年度第一學期 博士班資格考 試題卷

NATIONAL CHENGCHI UNIVERSITY EXAMINATION FORM

系別	應用數學系	考試科目	高等機率論	考試日期	**2023** 年 **9** 月 **18** 日	考試時間	**09:00** 至 **12:00**

注意事項
- 務必作答於答案卷並標明題號，請勿作答於試題卷上，否則不予計分。
- 本試題卷共有 **5** 個問題, 總計 **100** 分。

Notice: In the following, we use $\mathbb{N}$ to denote the set of positive integers and write $\mathbb{P}(E)$ and $\mathbb{E}X$ for the probabiltiy of event E and the expextation of random variable X. When writing $\mathbb{E}(X|Y)$, we mean the conditional expectation of random variable X given the σ-field generated by Y.

1. (20 %) For $n \in \mathbb{N}$, let X_n be a random variable having absolutely continuous distribution with density
$$f_n(x) = \begin{cases} c_n|\sin(n\pi x)| & \text{for } x \in (0,1), \\ 0 & \text{otherwise,} \end{cases}$$
where c_n is a constant.
   (1) (5%) Determine c_n.
   (2) (15%) Show that X_n converges in distribution to some random variable X and determine the distribution function of X.

2. (20 %) Let $(Y_n)_{n=1}^{\infty}$ be a sequence of i.i.d. random variables satisfying $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = 1/2$ for $n \in \mathbb{N}$ and set $Y = \displaystyle\sum_{n=1}^{\infty} \frac{Y_n}{2^n}$. Find the distribution of Y.

3. (20 %) Fix $c > 0$. Let $(T_n)_{n=1}^{\infty}$ be a sequence of independent random variables satisfying $\mathbb{P}(T_n = n^c) = \mathbb{P}(T_n = -n^c) = 1/2$ for $n \in \mathbb{N}$. Set $S_n = T_1 + T_2 + \cdots + T_n$. Show that $\dfrac{S_n}{n}$ converges to 0 almost surely if and only if $c < 1/2$.

4. (20 %) Let $(W_n)_{n=1}^{\infty}$ be i.i.d. random variables with $\mathbb{E}|W_1| < \infty$ and set $Z_n = W_1 + \cdots + W_n$. Show that $\mathbb{E}(W_1|Z_n, Z_{n+1}, \ldots) = \dfrac{Z_n}{n}$ almost surely.

5. (20 %) Let $(U_n)_{n=1}^{\infty}$ be a sequence of i.i.d. standard normal random variables and set
$$V_n = \frac{1}{\sqrt{n+1}} \exp\left\{ \frac{(U_1 + U_2 + \cdots + U_n)^2}{2(n+1)} \right\}.$$
   (1) (10%) Show that $(V_n)_{n=1}^{\infty}$ is a martingale.
   (2) (10%) Show that V_n converges almost surely and find the distribution function of the limit.